

# Forward and inverse solutions for 2D semi-infinite strip with self-equilibrated end loading

Anton Shterenlikht, Hokyeom Kim  
Mech Eng Dept, The University of Bristol, BS8 1TR, UK

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## 1 Basics

The method was first published in 2000 [1]. The method was applied to the measurement of res. stress in aluminium in [2, 3, 4].

Here more explicit details of the forward and inverse solutions are shown.

It's easy to check that the following stress functions:

$$f = C e^{-\gamma x_1/c} \left( \xi \cos \frac{\gamma x_2}{c} + \frac{\gamma x_2}{c} \sin \frac{\gamma x_2}{c} \right) \quad (1)$$

and

$$g = C e^{-\phi x_1/c} \left( \psi \sin \frac{\phi x_2}{c} + \frac{\phi x_2}{c} \cos \frac{\phi x_2}{c} \right) \quad (2)$$

are solutions to the biharmonic equation

$$f_{,iijj} = 0, \quad g_{,iijj} = 0 \quad (3)$$

(1) is an even function of  $x_2$  and (2) is an odd function of  $x_2$ .  $x_1$  is along the beam, with  $x_1 = 0$  being the boundary.  $2c$  is the width of the beam.  $x_2 = 0$  is the beam centre and neutral line, i.e.  $\epsilon_{22}(x_2 = 0) = 0$ . But generally  $u_2(x_2 = 0) \neq 0$ ! The top and bottom surfaces of the beam, i.e.  $x_2 = \pm c$  are free, i.e.  $\sigma_{22} = \sigma_{12} = 0$  there, see Fig. 1.

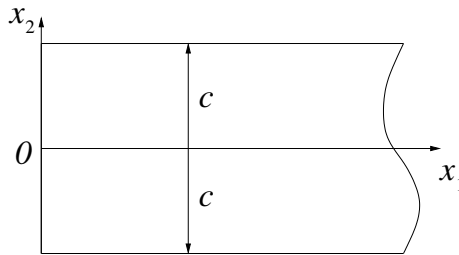


Figure 1: Schematic diagram of the semi-infinite strip.

or

From [5, pp. 61-62] these boundary conditions are satisfied if

$$f = \frac{\partial f}{\partial x_2} = 0 \quad \text{on} \quad x_2 = \pm c \quad (4)$$

because from these follows:

$$\frac{\partial^2 f}{\partial x_1^2} \equiv \sigma_{22} = 0; \quad -\frac{\partial^2 f}{\partial x_1 \partial x_2} \equiv \sigma_{12} = 0 \quad \text{on} \quad x_2 = \pm c \quad (5)$$

Note that

$$\frac{\partial f}{\partial x_1} \equiv \text{const} \cdot f \quad (6)$$

For  $g$  the BC are the same:

$$g = \frac{\partial g}{\partial x_2} = 0 \quad \text{on} \quad x_2 = \pm c \quad (7)$$

because from these follows:

$$\frac{\partial^2 g}{\partial x_1^2} \equiv \sigma_{22} = 0; \quad -\frac{\partial^2 g}{\partial x_1 \partial x_2} \equiv \sigma_{12} = 0 \quad \text{on} \quad x_2 = \pm c \quad (8)$$

In addition for each  $x_1 = \text{const}$ , including  $x_1 = 0$ , the total force along  $x_1$  and the total moment are zero:

$$\int_{-c}^{+c} \sigma_{11} dx_2 = 0 \quad ; \quad \int_{-c}^{+c} \sigma_{11} x_2 dx_2 = 0 \quad (9)$$

Satisfying the BC one obtains:

$$\sin 2\gamma + 2\gamma = 0 \quad (10)$$

and

$$\xi = -\gamma \tan \gamma \quad (11)$$

for the even function, and

$$\sin 2\phi - 2\phi = 0 \quad (12)$$

and

$$\psi = -\phi / \tan \phi \quad (13)$$

for the odd function.

Since both  $f$  and  $g$  satisfy the biharmonic equation, then their individual real and imaginary parts satisfy it as well. So using all  $\gamma$  and  $\phi$  roots, the stress function for the case of loading the  $x_1 = 0$  boundary by an arbitrary normal and shear stresses (subject to constraint that the total load in  $x_1$  is zero) will be written as infinite series:

$$\theta = \sum_{i=1}^{\infty} a_i \text{Re} f^i + b_i \text{Im} f^i + c_i \text{Re} g^i + d_i \text{Im} g^i \quad (14)$$

The roots for  $\gamma$  and  $\phi$  are complex conjugate and for each root, the minus root is also root. We are only looking at roots with positive real part, because these are going to decay with  $x_1$ . We neglect the conjugates because these would just introduce a factor of 2 into (14).

The first 3 roots for  $\gamma, \phi$  are:

```

! Using SLATEC version: 4.1
! Using DNSQ: http://netlib.org/slatec/src/dnsq.f
! xtol= 1.7763568394002505E-014
complex( kind=rk ), parameter :: gamma( 200 ) = (/ &
( 2.1061961152453303E+00_rk , 1.1253643058009302E+00_rk ) , & ! 1
( 5.3562686986396306E+00_rk , 1.5515743729126248E+00_rk ) , & ! 2
( 8.5366824265759149E+00_rk , 1.7755436735110401E+00_rk ) , & ! 3

complex( kind=rk ), parameter :: phi( 200 ) = (/ &
( 3.7488381388881926E+00_rk , 1.3843391414936608E+00_rk ) , & ! 1
( 6.9499798569882323E+00_rk , 1.6761049424267525E+00_rk ) , & ! 2
( 1.0119258853915010E+01_rk , 1.8583838398762496E+00_rk ) , & ! 3

```

Use `roots8.f90` to recalculate and to see the error in each solution.

## 2 Plane stress, tangential disp. fields

Using Hooke's law for plane stress,  $\sigma_{33} = 0$ :

$$\epsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22}) \quad ; \quad \epsilon_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{11}) \quad ; \quad \epsilon_{12} = \frac{1+\nu}{E}\sigma_{12} \quad (15)$$

and small strain theory:

$$\epsilon_{11} = u_{1,1} \quad ; \quad \epsilon_{22} = u_{2,2} \quad ; \quad \epsilon_{12} = 1/2(u_{1,2} + u_{2,1}) \quad (16)$$

the displacements are:

$$u_1 = \int \epsilon_{11} dx_1 + u_1^0 \quad (17)$$

$$u_2 = \int \epsilon_{22} dx_2 + u_2^0 \quad (18)$$

where  $u_1^0, u_2^0$  are known displacements at some point.

### 2.1 Integration limits along $x_1$ and $u_1^0$

The BC lead to  $u_1(x_1 = +\infty) = 0$ . Hence  $u_1^0 = 0$  and integration should be from  $+\infty$  to  $x_1$ :

$$u_1 = \int_{+\infty}^{x_1} \epsilon_{11} dx_1 \quad (19)$$

### 2.2 Integration limits along $x_2$ and $u_2^0$

$u_2^0$  is the displacement of point  $(x_1, 0)$  relative to point  $(+\infty, 0)$  along  $x_2$ . This can be achieved by integrating along path A in Fig. 2.2. Then integration is carried out along path B (Fig. 2.2), from  $x_2 = 0$ .

From (16)(c):

$$u_{2,1} = 2\epsilon_{12} - u_{1,2} \quad (20)$$

So that  $u_2^0$  can be calculated as follows:

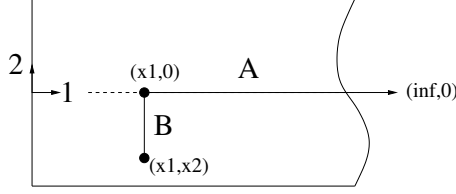


Figure 2: Integration path for  $u_2$ .

$$u_2^0 = \int_{+\infty}^{x_1} 2\epsilon_{12} dx_1 \Big|_{x_2=0} - \int_{+\infty}^{x_1} u_{1,2} dx_1 \Big|_{x_2=0} \quad (21)$$

so that (18) can be rewritten as

$$u_2 = \int_{+\infty}^{x_1} 2\epsilon_{12} dx_1 \Big|_{x_2=0} - \int_{+\infty}^{x_1} u_{1,2} dx_1 \Big|_{x_2=0} + \int_0^{x_2} \epsilon_{22} dx_2 \quad (22)$$

or using (19):

$$u_2 = \int_{+\infty}^{x_1} 2\epsilon_{12} dx_1 \Big|_{x_2=0} - \int_{+\infty}^{x_1} \int_{+\infty}^{x_1} \epsilon_{11,2} dx_1 dx_1 \Big|_{x_2=0} + \int_0^{x_2} \epsilon_{22} dx_2 \quad (23)$$

Let's express the first two integrals in 23 via  $\theta$ . From (15):

$$\epsilon_{12} = -\frac{1+\nu}{E} \theta_{,12} \quad (24)$$

$$\epsilon_{11,2} = \frac{1}{E} (\sigma_{11,2} - \nu \sigma_{22,2}) = \frac{1}{E} (\theta_{,222} - \nu \theta_{,112}) \quad (25)$$

where from (14)

$$\theta_{,11} = \sum_{i=1}^{\infty} a_i \text{Re} f_{,11}^i + b_i \text{Im} f_{,11}^i + c_i \text{Re} g_{,11}^i + d_i \text{Im} g_{,11}^i \quad (26)$$

$$\theta_{,22} = \sum_{i=1}^{\infty} a_i \text{Re} f_{,22}^i + b_i \text{Im} f_{,22}^i + c_i \text{Re} g_{,22}^i + d_i \text{Im} g_{,22}^i \quad (27)$$

$$\theta_{,12} = \sum_{i=1}^{\infty} a_i \text{Re} f_{,12}^i + b_i \text{Im} f_{,12}^i + c_i \text{Re} g_{,12}^i + d_i \text{Im} g_{,12}^i \quad (28)$$

Now from (1)-(2), assuming  $C = c^2$  (dropping subscript  $i$  for brevity):

$$f_{,11} = \gamma^2 \exp\left(-\frac{\gamma x_1}{c}\right) \left( \xi \cos \frac{\gamma x_2}{c} + \frac{\gamma x_2}{c} \sin \frac{\gamma x_2}{c} \right) \quad (29)$$

$$f_{,22} = \gamma^2 \exp\left(-\frac{\gamma x_1}{c}\right) \left( (2 - \xi) \cos \frac{\gamma x_2}{c} - \frac{\gamma x_2}{c} \sin \frac{\gamma x_2}{c} \right) \quad (30)$$

$$f_{,222} = \frac{\gamma^3}{c} \exp\left(-\frac{\gamma x_1}{c}\right) \left( (-3 + \xi) \sin \frac{\gamma x_2}{c} - \frac{\gamma x_2}{c} \cos \frac{\gamma x_2}{c} \right) \quad (31)$$

$$f_{,12} = \gamma^2 \exp\left(-\frac{\gamma x_1}{c}\right) \left( (\xi - 1) \sin \frac{\gamma x_2}{c} - \frac{\gamma x_2}{c} \cos \frac{\gamma x_2}{c} \right) \quad (32)$$

$$f_{,112} = -\frac{\gamma^3}{c} \exp\left(-\frac{\gamma x_1}{c}\right) \left( (\xi - 1) \sin \frac{\gamma x_2}{c} - \frac{\gamma x_2}{c} \cos \frac{\gamma x_2}{c} \right) \quad (33)$$

$$g_{,11} = \phi^2 \exp\left(-\frac{\phi x_1}{c}\right) \left( \psi \sin \frac{\phi x_2}{c} + \frac{\phi x_2}{c} \cos \frac{\phi x_2}{c} \right) \quad (34)$$

$$g_{,22} = \phi^2 \exp\left(-\frac{\phi x_1}{c}\right) \left( (-2 - \psi) \sin \frac{\phi x_2}{c} - \frac{\phi x_2}{c} \cos \frac{\phi x_2}{c} \right) \quad (35)$$

$$g_{,222} = \frac{\phi^3}{c} \exp\left(-\frac{\phi x_1}{c}\right) \left( (-3 - \psi) \cos \frac{\phi x_2}{c} + \frac{\phi x_2}{c} \sin \frac{\phi x_2}{c} \right) \quad (36)$$

$$g_{,12} = \phi^2 \exp\left(-\frac{\phi x_1}{c}\right) \left( (-1 - \psi) \cos \frac{\phi x_2}{c} + \frac{\phi x_2}{c} \sin \frac{\phi x_2}{c} \right) \quad (37)$$

$$g_{,112} = -\frac{\phi^3}{c} \exp\left(-\frac{\phi x_1}{c}\right) \left( (-1 - \psi) \cos \frac{\phi x_2}{c} + \frac{\phi x_2}{c} \sin \frac{\phi x_2}{c} \right) \quad (38)$$

So  $f_{,12}(x_2 = 0) = 0$  and

$$g_{,12}(x_2 = 0) = \phi^2 \exp\left(-\frac{\phi x_1}{c}\right) (-1 - \psi) = -(1 + \psi) \phi^2 \exp\left(-\frac{\phi x_1}{c}\right) \quad (39)$$

Let's introduce another function,  $q(x_1)$ , for  $g_{,12}(x_2 = 0)$ :

$$q(x_1) = -(1 + \psi) \phi^2 \exp\left(-\frac{\phi x_1}{c}\right) \quad (40)$$

so from (23), (24):

$$\int_{+\infty}^{x_1} 2\epsilon_{12} dx_1 \Big|_{x_2=0} = -\frac{2(1+\nu)}{E} \sum_{i=1}^{\infty} c_i \int_{+\infty}^{x_1} \operatorname{Re} q(x_1) dx_1 + d_i \int_{+\infty}^{x_1} \operatorname{Im} q(x_1) dx_1 \quad (41)$$

or, changing the limits of integration:

$$\int_{+\infty}^{x_1} 2\epsilon_{12} dx_1 \Big|_{x_2=0} = \frac{2(1+\nu)}{E} \sum_{i=1}^{\infty} c_i \int_{x_1}^{+\infty} \operatorname{Re} q(x_1) dx_1 + d_i \int_{x_1}^{+\infty} \operatorname{Im} q(x_1) dx_1 \quad (42)$$

For the second integral in (23) the necessary derivatives are:  $f_{,222}(x_2 = 0) = f_{,112}(x_2 = 0) = 0$  and

$$g_{,222}(x_2 = 0) = \frac{\phi^3}{c} \exp\left(-\frac{\phi x_1}{c}\right) (-3 - \psi) \quad (43)$$

$$g_{,112}(x_2 = 0) = \frac{\phi^3}{c} \exp\left(-\frac{\phi x_1}{c}\right) (1 + \psi) \quad (44)$$

so from (25):

$$\epsilon_{11,2}(x_2 = 0) = \frac{1}{E} \sum_{i=1}^{\infty} c_i \operatorname{Re}(g_{,222}^i - \nu g_{,112}^i) + d_i \operatorname{Im}(g_{,222}^i - \nu g_{,112}^i) \Big|_{x_2=0} \quad (45)$$

and

$$\begin{aligned}
& - \int_{+\infty}^{x_1} \int_{+\infty}^{x_1} \epsilon_{11,2} dx_1 dx_1 \Big|_{x_2=0} \\
& = - \frac{1}{E} \sum_{i=1}^{\infty} c_i \operatorname{Re} \int_{+\infty}^{x_1} \int_{+\infty}^{x_1} (g_{,222}^i - \nu g_{,112}^i) dx_1 dx_1 \Big|_{x_2=0} \\
& \quad - d_i \operatorname{Im} \int_{+\infty}^{x_1} \int_{+\infty}^{x_1} (g_{,222}^i - \nu g_{,112}^i) dx_1 dx_1 \Big|_{x_2=0} \quad (46)
\end{aligned}$$

or, (1) changing the integration limits twice and keeping the sign the same, and (2) moving the minus inside the integrand:

$$\begin{aligned}
& - \int_{+\infty}^{x_1} \int_{+\infty}^{x_1} \epsilon_{11,2} dx_1 dx_1 \Big|_{x_2=0} \\
& = \frac{1}{E} \sum_{i=1}^{\infty} c_i \operatorname{Re} \int_{x_1}^{+\infty} \int_{x_1}^{+\infty} (\nu g_{,112}^i - g_{,222}^i) dx_1 dx_1 \Big|_{x_2=0} \\
& \quad + d_i \operatorname{Im} \int_{x_1}^{+\infty} \int_{x_1}^{+\infty} (\nu g_{,112}^i - g_{,222}^i) dx_1 dx_1 \Big|_{x_2=0} \quad (47)
\end{aligned}$$

where from (43), (44):

$$\nu g_{,112}^i - g_{,222}^i \Big|_{x_2=0} = \frac{\phi^3}{c} \exp\left(-\frac{\phi x_1}{c}\right) (\nu(1 + \psi) + 3 + \psi) \quad (48)$$

$c$  can be moved from (48) to (47):

$$\begin{aligned}
& - \int_{+\infty}^{x_1} \int_{+\infty}^{x_1} \epsilon_{11,2} dx_1 dx_1 \Big|_{x_2=0} \\
& = \frac{1}{cE} \sum_{i=1}^{\infty} c_i \operatorname{Re} \int_{x_1}^{+\infty} \int_{x_1}^{+\infty} z(x_1) dx_1 dx_1 \\
& \quad + d_i \operatorname{Im} \int_{x_1}^{+\infty} \int_{x_1}^{+\infty} z(x_1) dx_1 dx_1 \quad (49)
\end{aligned}$$

where

$$z(x_1) = \phi^3 \exp\left(-\frac{\phi x_1}{c}\right) (\nu(1 + \psi) + 3 + \psi) \quad (50)$$

Now from (23):

$$u_2 = \int_0^{x_2} \epsilon_{22} dx_2 + s(x_1) \quad (51)$$

where  $s(x_1)$  is the function giving the motion of points initially on  $x_2 = 0$ , i.e. the central line, along  $x_2$ :

$$s(x_1) = \int_{+\infty}^{x_1} 2\epsilon_{12} dx_1 \Big|_{x_2=0} - \int_{+\infty}^{x_1} \int_{+\infty}^{x_1} \epsilon_{11,2} dx_1 dx_1 \Big|_{x_2=0} \quad (52)$$

Note that from calculations, for non-symm. loading,  $s(x_1)$  is quite large. Neglecting this correction in inverse solution can cause large errors.

### 2.3 Full expressions for displacements

From (15), (19) and (51):

$$u_1 = \frac{1}{E} \int_{+\infty}^{x_1} (\sigma_{11} - \nu\sigma_{22}) dx_1 \quad (53)$$

$$u_2 = \frac{1}{E} \int_0^{x_2} (\sigma_{22} - \nu\sigma_{11}) dx_2 + s(x_1) \quad (54)$$

or

$$u_1 = \frac{1}{E} \int_{+\infty}^{x_1} (\theta_{,22} - \nu\theta_{,11}) dx_1 \quad (55)$$

$$u_2 = \frac{1}{E} \int_0^{x_2} (\theta_{,11} - \nu\theta_{,22}) dx_2 + s(x_1) \quad (56)$$

where

$$\begin{aligned} (\theta_{,22} - \nu\theta_{,11}) = & \sum_{i=1}^{\infty} a_i \text{Re}(f_{,22}^i - \nu f_{,11}^i) + b_i \text{Im}(f_{,22}^i - \nu f_{,11}^i) \\ & + c_i \text{Re}(g_{,22}^i - \nu g_{,11}^i) + d_i \text{Im}(g_{,22}^i - \nu g_{,11}^i) \end{aligned} \quad (57)$$

$$\begin{aligned} (\theta_{,11} - \nu\theta_{,22}) = & \sum_{i=1}^{\infty} a_i \text{Re}(f_{,11}^i - \nu f_{,22}^i) + b_i \text{Im}(f_{,11}^i - \nu f_{,22}^i) \\ & + c_i \text{Re}(g_{,11}^i - \nu g_{,22}^i) + d_i \text{Im}(g_{,11}^i - \nu g_{,22}^i) \end{aligned} \quad (58)$$

or

$$\begin{aligned} Eu_1 = & \sum_{i=1}^{\infty} a_i \int_{+\infty}^{x_1} \text{Re}(f_{,22}^i - \nu f_{,11}^i) dx_1 + b_i \int_{+\infty}^{x_1} \text{Im}(f_{,22}^i - \nu f_{,11}^i) dx_1 \\ & + c_i \int_{+\infty}^{x_1} \text{Re}(g_{,22}^i - \nu g_{,11}^i) dx_1 + d_i \int_{+\infty}^{x_1} \text{Im}(g_{,22}^i - \nu g_{,11}^i) dx_1 \end{aligned} \quad (59)$$

$$\begin{aligned} Eu_2 = & \sum_{i=1}^{\infty} a_i \int_0^{x_2} \text{Re}(f_{,11}^i - \nu f_{,22}^i) dx_2 + b_i \int_0^{x_2} \text{Im}(f_{,11}^i - \nu f_{,22}^i) dx_2 \\ & + c_i \int_0^{x_2} \text{Re}(g_{,11}^i - \nu g_{,22}^i) dx_2 + d_i \int_0^{x_2} \text{Im}(g_{,11}^i - \nu g_{,22}^i) dx_2 + Es(x_1) \end{aligned} \quad (60)$$

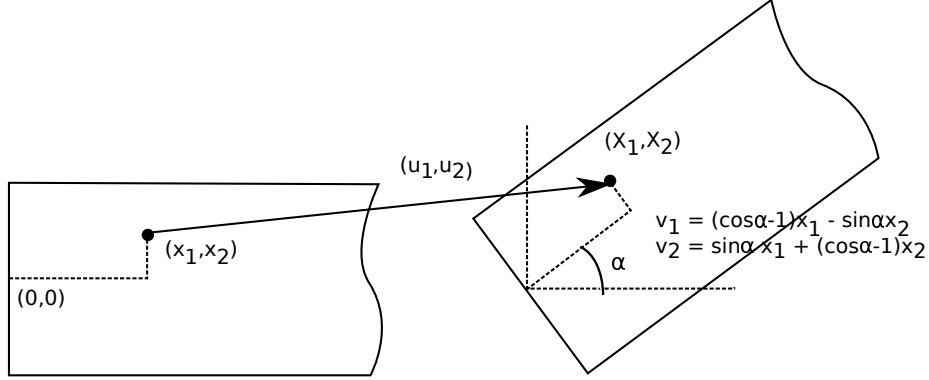


Figure 3: Schematic of rigid body motion. Displacements due to rigid body motion can be expressed as translation,  $\mathbf{u}^T$  and that due to rotation by angle  $\alpha$ ,  $\mathbf{u}^R$ .

where from (29)-(37):

$$f_{,22}^i - \nu f_{,11}^i = \gamma^2 \exp\left(-\frac{\gamma x_1}{c}\right) \left[ (2 - \xi - \nu\xi) \cos \frac{\gamma x_2}{c} - (1 + \nu) \frac{\gamma x_2}{c} \sin \frac{\gamma x_2}{c} \right] \quad (61)$$

$$f_{,11}^i - \nu f_{,22}^i = \gamma^2 \exp\left(-\frac{\gamma x_1}{c}\right) \left[ (\xi - \nu(2 - \xi)) \cos \frac{\gamma x_2}{c} + (1 + \nu) \frac{\gamma x_2}{c} \sin \frac{\gamma x_2}{c} \right] \quad (62)$$

$$g_{,22}^i - \nu g_{,11}^i = \phi^2 \exp\left(-\frac{\phi x_1}{c}\right) \left[ (-2 - \psi - \nu\psi) \sin \frac{\phi x_2}{c} - (1 + \nu) \frac{\phi x_2}{c} \cos \frac{\phi x_2}{c} \right] \quad (63)$$

$$g_{,11}^i - \nu g_{,22}^i = \phi^2 \exp\left(-\frac{\phi x_1}{c}\right) \left[ (\psi + \nu(2 + \psi)) \sin \frac{\phi x_2}{c} + (1 + \nu) \frac{\phi x_2}{c} \cos \frac{\phi x_2}{c} \right] \quad (64)$$

## 2.4 Rigid body motion

Displacement experiments might suffer from rigid body motion. This section analysiss in-plane rigid body motion and adds its contribution to the measured displacements at all points.

Rigid body motion creates displacement  $\mathbf{u}^M$  at all points:

$$\begin{Bmatrix} u_1^M \\ u_2^M \end{Bmatrix} = \begin{Bmatrix} u_1^T \\ u_2^T \end{Bmatrix} + \begin{Bmatrix} u_1^R \\ u_2^R \end{Bmatrix} \quad (65)$$

where  $\mathbf{u}^T$  is due to in-plane translation, and where  $\mathbf{u}^R$  is due to in-plane rotation by angle  $\alpha$ .  $\mathbf{u}^R$  can found as the difference between coordinates of point P after rotation,  $\mathbf{x}^R$ , and its original coordinates (before rotation),  $\mathbf{x}$ :



$$\begin{Bmatrix} u_1^R \\ u_2^R \end{Bmatrix} = \begin{Bmatrix} x_1^R - x_1 \\ x_1^R - x_2 \end{Bmatrix} \quad (66)$$

where

$$\begin{Bmatrix} x_1^R \\ x_2^R \end{Bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \cos \alpha x_1 - \sin \alpha x_2 \\ \sin \alpha x_1 + \cos \alpha x_2 \end{Bmatrix} \quad (67)$$

Substituting (67), (66) into (65):

$$\begin{Bmatrix} u_1^M \\ u_2^M \end{Bmatrix} = \begin{Bmatrix} u_1^T + (\cos \alpha - 1)x_1 - \sin \alpha x_2 \\ u_2^T + \sin \alpha x_1 + (\cos \alpha - 1)x_2 \end{Bmatrix} \quad (68)$$

Displacements due to rigid body motion are added to (55) and (56):

$$u_1 = \frac{1}{E} \left( \int_{+\infty}^{x_1} (\theta_{,22} - \nu \theta_{,11}) dx_1 \right) + u_1^M \quad (69)$$

$$u_2 = \frac{1}{E} \left( \int_0^{x_2} (\theta_{,11} - \nu \theta_{,22}) dx_2 + s(x_1) \right) + u_2^M \quad (70)$$

or from (59) - (60):

$$\begin{aligned} Eu_1 = & \sum_{i=1}^{\infty} a_i \int_{+\infty}^{x_1} \operatorname{Re}(f_{,22}^i - \nu f_{,11}^i) dx_1 + b_i \int_{+\infty}^{x_1} \operatorname{Im}(f_{,22}^i - \nu f_{,11}^i) dx_1 \\ & + c_i \int_{+\infty}^{x_1} \operatorname{Re}(g_{,22}^i - \nu g_{,11}^i) dx_1 + d_i \int_{+\infty}^{x_1} \operatorname{Im}(g_{,22}^i - \nu g_{,11}^i) dx_1 \\ & + E(u_1^T + (\cos \alpha - 1)x_1 - \sin \alpha x_2) \quad (71) \end{aligned}$$

$$\begin{aligned} Eu_2 = & \sum_{i=1}^{\infty} a_i \int_0^{x_2} \operatorname{Re}(f_{,11}^i - \nu f_{,22}^i) dx_2 + b_i \int_0^{x_2} \operatorname{Im}(f_{,11}^i - \nu f_{,22}^i) dx_2 \\ & + c_i \int_0^{x_2} \operatorname{Re}(g_{,11}^i - \nu g_{,22}^i) dx_2 + d_i \int_0^{x_2} \operatorname{Im}(g_{,11}^i - \nu g_{,22}^i) dx_2 + Es(x_1) \\ & + E(u_2^T + \sin \alpha x_1 + (\cos \alpha - 1)x_2) \quad (72) \end{aligned}$$

Now we really have to unwrap  $s(x_1)$  to separate terms with different series expansion coefficients. From (42), (49), (52):

$$\begin{aligned} Es(x_1) = & 2(1 + \nu) \sum_{i=1}^{\infty} c_i \int_{x_1}^{+\infty} \operatorname{Re}q(x_1) dx_1 + d_i \int_{x_1}^{+\infty} \operatorname{Im}q(x_1) dx_1 \\ & + \frac{1}{c} \sum_{i=1}^{\infty} c_i \int_{x_1}^{+\infty} \int_{x_1}^{+\infty} \operatorname{Re}z(x_1) dx_1 dx_1 \\ & + d_i \int_{x_1}^{+\infty} \int_{x_1}^{+\infty} \operatorname{Im}z(x_1) dx_1 dx_1 \quad (73) \end{aligned}$$

or grouping together terms with the same series expansion coefficients:

$$Es(x_1) = \sum_{i=1}^{\infty} c_i \left\{ 2(1+\nu) \int_{x_1}^{+\infty} \operatorname{Re}q(x_1) dx_1 + \frac{1}{c} \int_{x_1}^{+\infty} \int_{x_1}^{+\infty} \operatorname{Re}z(x_1) dx_1 dx_1 \right\} \\ + d_i \left\{ 2(1+\nu) \int_{x_1}^{+\infty} \operatorname{Im}q(x_1) dx_1 + \frac{1}{c} \int_{x_1}^{+\infty} \int_{x_1}^{+\infty} \operatorname{Im}z(x_1) dx_1 dx_1 \right\} \quad (74)$$

so that (72) can be rewritten as:

$$Eu_2 = \sum_{i=1}^{\infty} a_i \int_0^{x_2} \operatorname{Re}(f_{,11}^i - \nu f_{,22}^i) dx_2 + b_i \int_0^{x_2} \operatorname{Im}(f_{,11}^i - \nu f_{,22}^i) dx_2 \\ + c_i \left\{ \int_0^{x_2} \operatorname{Re}(g_{,11}^i - \nu g_{,22}^i) dx_2 + 2(1+\nu) \int_{x_1}^{+\infty} \operatorname{Re}q(x_1) dx_1 + \frac{1}{c} \int_{x_1}^{+\infty} \int_{x_1}^{+\infty} \operatorname{Re}z(x_1) dx_1 dx_1 \right\} \\ + d_i \left\{ \int_0^{x_2} \operatorname{Im}(g_{,11}^i - \nu g_{,22}^i) dx_2 + 2(1+\nu) \int_{x_1}^{+\infty} \operatorname{Im}q(x_1) dx_1 + \frac{1}{c} \int_{x_1}^{+\infty} \int_{x_1}^{+\infty} \operatorname{Im}z(x_1) dx_1 dx_1 \right\} \\ + E(u_2^T + \sin \alpha x_1 + (\cos \alpha - 1)x_2) \quad (75)$$

### 3 Stress

$$\sigma_{11} = \sum_{i=1}^N a_i \operatorname{Re}f_{,22} + b_i \operatorname{Im}f_{,22} + c_i \operatorname{Re}g_{,22} + d_i \operatorname{Im}g_{,22} \quad (76)$$

$$\sigma_{22} = \sum_{i=1}^N a_i \operatorname{Re}f_{,11} + b_i \operatorname{Im}f_{,11} + c_i \operatorname{Re}g_{,11} + d_i \operatorname{Im}g_{,11} \quad (77)$$

$$\sigma_{12} = \sum_{i=1}^N -a_i \operatorname{Re}f_{,12} - b_i \operatorname{Im}f_{,12} - c_i \operatorname{Re}g_{,12} - d_i \operatorname{Im}g_{,12} \quad (78)$$

For stress on the boundary set  $x_1 = 0$  in (29)-(37).

### 4 Forward problem

By setting  $a_i, b_i, c_i, d_i$  the problem is fully constrained. Stresses are calculated immediately from (76)–(78) and displacements from (59), (60).

One can set all  $a_i$  and  $b_i$  at random. Since these relate to  $f$ , these coefficients set the symmetric stress fields which are always in momentum equilibrium.

However, one cannot set all  $c_i$  and  $d_i$  at random, because this will violate the momentum equilibrium:

$$\int_{-c}^c \sigma_{11} x_2 dx_2 = 0 \quad (79)$$

or, using (76):

$$\int_{-c}^c \left( \sum_{i=1}^N c_i \operatorname{Re}g_{,22} + d_i \operatorname{Im}g_{,22} \right) x_2 dx_2 = 0 \quad (80)$$

Assuming one sets all  $c_i, i = 1, \dots, N$ , and all but last  $d_i$ , i.e.  $d_i, i = 1, \dots, N-1$ , then  $d_N$  is found as follows:

$$\sum_{i=1}^N c_i \int_{-c}^c \text{Reg}_{,22} x_2 dx_2 + \sum_{i=1}^{N-1} d_i \int_{-c}^c \text{Img}_{,22} x_2 dx_2 = -d_N \int_{-c}^c \text{Img}_{,22} x_2 dx_2 \quad (81)$$

Note that all integrals are very small.  $d_N$  is found by a division of a small numerator over a small denominator. This might cause loss of precision. Care should be taken, perhaps a check, to make sure (79) holds numerically.

## 5 Units

From (5) and (8) the units of  $f, g, \theta$  is force. Let's assume that the unit of force is N and the unit of length is mm. One must use the same units for  $c$  as for coordinates  $x_i$ . In this case, and because  $\gamma, \phi, \xi$  and  $\psi$  are dimensionless, the units  $C$  are N. The units of (61)-(64) are MPa. The integrals in (59)-(60) are thus in MPa mm. Therefore  $a_i, b_i, c_i, d_i$  are dimensionless. The units for  $E$  must be MPa. Stress will be in MPa as well.

## 6 Integration limits and constants

The limits in (59), (60) could be swapped, with the sign changed, if the library used for calculation requires the lower limit be smaller than the upper limit.

The tests show that QUADPACK [6], at least DQAGSE and DQNG, correctly calculate integrals where the upper limit is less than the lower limit. However, it's easier to change the limits to make sure that the lower is always smaller than the upper and not to worry about this potential issue.

## 7 Inverse problem

Now assuming we know  $u_1, u_2$ , let's find  $\theta$ .

### 7.1 Displacement of the centre line ( $x_2 = 0$ )

Since we are not guaranteed to have any data points with  $x_2 = 0$ , we need to interpolate to calculate  $u_2(x_2 = 0)$ . I use here DTENSBS sublibrary in CMLIB package.

### 7.2 Matrix form

Let's rewrite (71)-(72) in matrix form. The notation is changed to match Lapack. Let's assume we collect data from  $p$  points. We have 2 displacement values per point. So in total we have  $M = 2p$  displacement values. We arrange them in a vector  $\mathbf{B}$  as follows:

$$\mathbf{B} = E \begin{Bmatrix} u_1^1 \\ u_2^1 \\ \vdots \\ u_1^p \\ u_2^1 \\ u_2^2 \\ \vdots \\ u_2^p \end{Bmatrix}_{M \times 1} \quad (82)$$

where the subscript denotes coordinate (1 or 2) and the superscript denotes the node number (1, 2, ..., p). Note that the vertical motion of the central line is subtracted from all  $u_2$ .

The upper series limit is capped at  $N$ . There are  $4N$  unknowns  $a_i, b_i, c_i, d_i$ . These are put into a vector, with factors  $\gamma^2$  and  $\phi^2$ . There are also 4 rigid body unknowns -  $u_1^T, u_2^T, \cos \alpha - 1$  and  $\sin \alpha$ . Note that although there are only 3 in-plane rigid body degrees of freedom, four unknowns are required for expressing these in a matrix equation:

$$\mathbf{x} = \begin{Bmatrix} a_1(\text{Re}\gamma_1)^2 \\ b_1(\text{Re}\gamma_1)^2 \\ c_1(\text{Re}\phi_1)^2 \\ d_1(\text{Re}\phi_1)^2 \\ \vdots \\ a_N(\text{Re}\gamma_N)^2 \\ b_N(\text{Re}\gamma_N)^2 \\ c_N(\text{Re}\phi_N)^2 \\ d_N(\text{Re}\phi_N)^2 \\ Eu_1^T \\ Eu_2^T \\ E(\cos \alpha - 1) \\ E \sin \alpha \end{Bmatrix}_{4(N+1) \times 1} \quad (83)$$

Note that we included certain parameters in the vector of unknowns, to make the matrix better balanced:

$$\mathbf{A} = \begin{bmatrix} A_1^1 & B_1^1 & C_1^1 & D_1^1 & \dots & A_N^1 & B_N^1 & C_N^1 & D_N^1 & 1 & 0 & x_1^1 & -x_2^1 \\ A_1^2 & B_1^2 & C_1^2 & D_1^2 & \dots & A_N^2 & B_N^2 & C_N^2 & D_N^2 & 1 & 0 & x_1^2 & -x_2^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_1^p & B_1^p & C_1^p & D_1^p & \dots & A_N^p & B_N^p & C_N^p & D_N^p & 1 & 0 & x_1^p & -x_2^p \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ E_1^1 & F_1^1 & G_1^1 & H_1^1 & \dots & E_N^1 & F_N^1 & G_N^1 & H_N^1 & 0 & 1 & x_2^1 & x_1^1 \\ E_1^2 & F_1^2 & G_1^2 & H_1^2 & \dots & E_N^2 & F_N^2 & G_N^2 & H_N^2 & 0 & 1 & x_2^2 & x_1^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ E_1^p & F_1^p & G_1^p & H_1^p & \dots & E_N^p & F_N^p & G_N^p & H_N^p & 0 & 1 & x_2^p & x_1^p \end{bmatrix}_{M \times 4(N+1)} \quad (84)$$

where  $x_1$  and  $x_2$  are the coordinates, the subscript in  $A, B, C, D$  denotes the number of the series expansion term, the superscript is the node number  $(1, 2, \dots, p)$  and all coefficients are as in (59)-(60):

$$A_i^j = -\frac{1}{(\operatorname{Re}\gamma_i)^2} \int_{x_1^j}^{+\infty} \operatorname{Re}(f_{,22}^i - \nu f_{,11}^i) dx_1 \quad B_i^j = -\frac{1}{(\operatorname{Re}\gamma_i)^2} \int_{x_1^j}^{+\infty} \operatorname{Im}(f_{,22}^i - \nu f_{,11}^i) dx_1 \quad (85)$$

$$C_i^j = -\frac{1}{(\operatorname{Re}\phi_i)^2} \int_{x_1^j}^{+\infty} \operatorname{Re}(g_{,22}^i - \nu g_{,11}^i) dx_1 \quad D_i^j = -\frac{1}{(\operatorname{Re}\phi_i)^2} \int_{x_1^j}^{+\infty} \operatorname{Im}(g_{,22}^i - \nu g_{,11}^i) dx_1 \quad (86)$$

$$E_i^j = \frac{1}{(\operatorname{Re}\gamma_i)^2} \int_0^{x_2^j} \operatorname{Re}(f_{,11}^i - \nu f_{,22}^i) dx_2 \quad F_i^j = \frac{1}{(\operatorname{Re}\gamma_i)^2} \int_0^{x_2^j} \operatorname{Im}(f_{,11}^i - \nu f_{,22}^i) dx_2 \quad (87)$$

$$G_i^j = \frac{1}{(\operatorname{Re}\phi_i)^2} \left\{ \int_0^{x_2^j} \operatorname{Re}(g_{,11}^i - \nu g_{,22}^i) dx_2 + 2(1 + \nu) \int_{x_1}^{+\infty} \operatorname{Re}q(x_1) dx_1 + \frac{1}{c} \int_{x_1}^{+\infty} \int_{x_1}^{+\infty} \operatorname{Re}z(x_1) dx_1 dx_1 \right\} \quad (88)$$

$$H_i^j = \frac{1}{(\operatorname{Re}\phi_i)^2} \left\{ \int_0^{x_2^j} \operatorname{Im}(g_{,11}^i - \nu g_{,22}^i) dx_2 + 2(1 + \nu) \int_{x_1}^{+\infty} \operatorname{Im}q(x_1) dx_1 + \frac{1}{c} \int_{x_1}^{+\infty} \int_{x_1}^{+\infty} \operatorname{Im}z(x_1) dx_1 dx_1 \right\} \quad (89)$$

The problem can be solved in standard LLS form:

$$\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{B}\|_2 \quad (90)$$

However, this is typically an ill-conditioned problem, and some stabilization is usually required. One can use truncated SVD or Tikhonov regularization [7]. Both techniques are available in Lapack.

If matrix  $\mathbf{A}$  is  $M \times 4(N+1)$ ,  $M > 4(N+1)$ , and of full rank, then the eigenvalues of  $\mathbf{A}$  are

$$s_1 \geq s_2 \geq \dots \geq s_{4(N+1)} \quad (91)$$

By setting the limit,  $r$ , on how small an eigenvalue can be, SVD can be truncated, i.e. set all  $s_i = 0$  if  $s_i < r s_1$ . This makes  $\mathbf{A}$  rank deficient. In this case we find the minimum norm solution, i.e. in addition to (90) we find

$$\min_{\mathbf{x}} \|\mathbf{x}\|_2 \quad (92)$$

The solution of (90) together with (92) is unique.

Finally, if it is clear that the solution is symmetric, then it's best to set all  $c_i, d_i$  to zero. Similarly, if the solution is clearly anti-symmetric, it's best to set all  $a_i, b_i$  to zero.

After  $\mathbf{x}$  is found,  $a_i \dots d_i$ , and  $u_1^T, u_2^T$  and  $\alpha$  are extracted from (83).

### 7.3 Symmetric problems

For pure symmetric problems stress function  $g$  is not used. Hence  $\mathbf{x}$  and  $\mathbf{M}$  can be simplified as:

$$\mathbf{x} = \begin{pmatrix} a_1(\text{Re}\gamma_1)^2 \\ b_1(\text{Re}\gamma_1)^2 \\ \vdots \\ a_N(\text{Re}\gamma_N)^2 \\ b_N(\text{Re}\gamma_N)^2 \\ Eu_1^T \\ Eu_2^T \\ E(\cos \alpha - 1) \\ E \sin \alpha \end{pmatrix}_{(2N+4) \times 1} \quad (93)$$

$$\mathbf{A} = \begin{bmatrix} A_1^1 & B_1^1 & \dots & A_N^1 & B_N^1 & 1 & 0 & x_1^1 & -x_2^1 \\ A_1^2 & B_1^2 & \dots & A_N^2 & B_N^2 & 1 & 0 & x_1^2 & -x_2^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_1^p & B_1^p & \dots & A_N^p & B_N^p & 1 & 0 & x_1^p & -x_2^p \\ E_1^1 & F_1^1 & \dots & E_N^1 & F_N^1 & 0 & 1 & x_2^1 & x_1^1 \\ E_1^2 & F_1^2 & \dots & E_N^2 & F_N^2 & 0 & 1 & x_2^2 & x_1^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ E_1^p & F_1^p & \dots & E_N^p & F_N^p & 0 & 1 & x_2^p & x_1^p \end{bmatrix}_{\mathbf{M} \times (2N+4)} \quad (94)$$

### 7.4 Anti-symmetric problems

For pure anti-symmetric problems  $f$  is not used.  $\mathbf{x}$  and  $\mathbf{M}$  can be simplified as:

$$\mathbf{x} = \begin{pmatrix} c_1(\text{Re}\phi_1)^2 \\ d_1(\text{Re}\phi_1)^2 \\ \vdots \\ c_N(\text{Re}\phi_N)^2 \\ d_N(\text{Re}\phi_N)^2 \\ Eu_1^T \\ Eu_2^T \\ E(\cos \alpha - 1) \\ E \sin \alpha \end{pmatrix}_{(2N+4) \times 1} \quad (95)$$

$$\mathbf{A} = \begin{bmatrix} C_1^1 & D_1^1 & \dots & C_N^1 & D_N^1 & 1 & 0 & x_1^1 & -x_2^1 \\ C_1^2 & D_1^2 & \dots & C_N^2 & D_N^2 & 1 & 0 & x_1^2 & -x_2^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ C_1^p & D_1^p & \dots & C_N^p & D_N^p & 1 & 0 & x_1^p & -x_2^p \\ G_1^1 & H_1^1 & \dots & G_N^1 & H_N^1 & 0 & 1 & x_2^1 & x_1^1 \\ G_1^2 & H_1^2 & \dots & G_N^2 & H_N^2 & 0 & 1 & x_2^2 & x_1^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ G_1^p & H_1^p & \dots & G_N^p & H_N^p & 0 & 1 & x_2^p & x_1^p \end{bmatrix}_{M \times (2N+4)} \quad (96)$$

## 8 Programs

There are a number of forward and inverse programs in the distribution. Shared and/or dynamic executables can be built on a number of platforms. Shared executables are given names ending in `.sx`. Dynamic executables are given names ending in `.dx`.

### 8.1 Forward solvers

#### 8.1.1 exact

Program `exact` solves the forward problem, i.e. calculating displacements and stresses everywhere from the given series expansion coefficients, or generating the series expansion coefficients at random. The main purpose of the program is to generate validation test cases, i.e. displacement fields for use with inverse programs, and corresponding stress fields.

The program takes the name of the settings file as an input. This is a plain text file with exactly 13 lines. A minimum of 2 fields per line are expected.

The first field is a character string with no spaces or tabs. This is the name of the input parameter. The value of this string is not used in the program. Hence any value can be assigned to the first field, to help the user give a useful identifier.

The second field is interpreted as a `REAL`, of a certain program specific kind. The value of this field is used in the program.

Any other fields on the line are ignored. These can be used to provide a comment to better label the values in the second field.

An example settings file is shown below.

```

c      70.0    half width, in mm
x1min  0.0     as a fraction of c
x1max  1.0     as a fraction of c
x2min  -1.0    as a fraction of c
x2max  1.0     as a fraction of c
dx1    0.05    as a fraction of c
dx2    0.05    as a fraction of c
noise  0.0     noise magnitude, as in u = u * ( 1 + RND*noise )
angle  0.123   in degrees, rigid body rotation about origin
tran1  0.6     in mm, rigid body translation along 1
tran2  -0.4    in mm, rigid body translation along 2

```

```
young 2.0e5 Young's modulus
poisson 0.33 Poisson's ratio
```

The settings file sets the elastic properties, the geometry of the problem, the extent of the field where displacements and stresses are calculated, the spatial sampling frequency within this field, the rigid body translation and rotation (to simulate the rigid body motion in experiment) and the magnitude of random noise to add to calculated displacements.

After reading the setting file, the program will ask the user whether to generate the coefficients at random, or read from file. If the user selects the option to read from file, then a file name must be entered. This is a plain text file with one line per term in the series. Each line has 4 fields separated by spaces or tabs. Each field is a `REAL` number. The first field is read in as  $a_i$ , the second as  $b_i$ , the third as  $c_i$  and the final fourth field is read in as  $d_i$ . The first line is term 1, the second line, if any, is term 2, etc. The number of lines in the file (terms) is not limited. The program will read the file until the end. An example coefficients file is shown below.

```
2.348133056E+00 4.096870272E+00 0.0e0 0.0e0
7.158751031E-01 4.758107668E-01 0.0e0 0.0e0
2.209624893E-01 1.262096797E-01 0.0e0 0.0e0
```

In this file there are 4 first terms in the series expansion. This is a symmetric loading, because all  $c_i = d_i = 0$ .

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